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Harmonic Analysis on the Space-Time Gauge Continuum

David D. Bleecker

Department of Mathematics, University of Hawaii

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The classical Kaluza–Klein unified field theory has previously been extended to unify and geometrize gravitational and gauge fields, through a study of the geometry of a bundle space P over space-time. Here, we examine the physical relevance of the Laplace operator on the complex-valued functions on P. The spectrum and eigenspaces are shown (via the Peter–Weyl theorem) to determine the possible masses of any type of particle field. In the Euclidean case, we prove that zero-mass particles necessarily come in infinite families. Also, lower bounds on masses of particles of a given type are obtained in terms of the curvature of P.

INTRODUCTION

Let $\pi: P \to M$ be a C^∞ principal bundle with group G, having Lie algebra $\mathcal{G}$. Suppose M has a metric tensor h_M, and let ω be a $\mathcal{G}$-valued connection 1-form (or gauge potential) on P. If k is some $\mathfrak{Ad}$-invariant inner product on $\mathcal{G}$, then we define a metric tensor h on P by $h(X,Y)= h_M(\pi_* X, \pi_* Y)+k(\omega(X),\omega(Y))$, where $X, Y \in T_p P \equiv$ tangent space of P at $p \in P$. For a full introduction to the above concepts, the reader may consult Bleecker (1981), Kabayashi and Nomizu (1963), Trautman (1980), etc. The physical significance of the geometry of (P, h) is well established in the case where (M, h_M) is a space-time and $G = U(1)$. Indeed, (P, h) is then the five-dimensional space in the classical Kaluza–Klein unified field theory (Klein, 1926). There are two significant facts. The Einstein field equations and Maxwell's equations result by equating to zero the first variation (with respect to h_M and ω) of the total scalar curvature of (P, h). Also, the geodesics of (P, h) project (via π) onto the space-time paths of charged particles on M where the charge is essentially the vertical component of the tangent vector of the geodesic on P. Both of these facts have appropriate generalizations to the case of an arbitrary compact group G, except that

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Maxwell's equations are replaced by the more general Yang–Mills equations, and charge must be understood in a generalized sense (e.g., isospin, hypercharge, color charge, weak charge, etc., depending on G); see Bleecker (1981), Cho (1975), Trautman (1980), etc.

In this paper, we concentrate on the relevance of the spectrum of the Laplace operator Δ of (P, h) on $C^\infty(P,\mathbb{C}) \equiv \mathbb{C}$-valued C^∞ functions on P. To avoid the difficulties that arise when h is not positive definite (e.g., nonellipticity of Δ) and when P is noncompact, we will assume henceforth that (M, h_M) is a compact connected Riemannian manifold as in Euclidean field theory, and that G is compact with k on $\mathcal{G}$ positive definite. In order that the spectrum of Δ be nonnegative, we define [for $u \in \mathbb{C}^\infty(P, C)$ and $p \in P$] $(\Delta u)(p)$ as *minus* the sum of the second derivatives at p of u along a frame of geodesics (i.e., with orthonormal tangent vectors) passing through p. For a given unitary representation r: $G \to U(W_r)$ there is also a Laplace operator Δ_r on the space $C(P, W_r) \equiv \{f\colon P \to W_r \mid f(pg) = r(g^{-1})f(p)$ for all $p \in P$; and f is $C^\infty\}$ of particle fields associated to r. In Section 3, we define Δ_r in terms of covariant differentiation (relative to ω) and its dual codifferential. The (Euclidean) mass^2 spectrum for particles arising from r is $\mathrm{Spec}(\Delta_r) \equiv \{m \in \mathbb{R} \mid \Delta_r f = mf$ for some $0 \neq f \in C(P, W_r)\}$. In Section 4, we prove that $\mathrm{Spec}(\Delta_r)$ for any r can be completely determined from $\mathrm{Spec}(\Delta) \equiv \{\lambda \in \mathbb{R} \mid \Delta u = \lambda u$ for some $0 \neq u \in C^\infty(P, \mathbb{C})\}$ and a knowledge of how the eigenspaces of Δ decompose into irreducible subspaces under the isometric action of G on (P, h).

In Section 4, we prove that if $0 \in \mathrm{Spec}(\Delta_r)$ for a nontrivial r, then the holonomy group G_0 of ω is not equal to G. Assuming also that G is connected and letting G_0' be the closure of G_0, then we prove that for each of the infinitely many eigenvalues of the Laplace operator for G/G_0' there corresponds a different irreducible unitary representation s of G such that $0 \in \mathrm{Spec}(\Delta_s)$.

In Section 5, we find that if the curvature (i.e., field strength) Ω of ω satisfies a certain nondegeneracy requirement, then there is a positive lower bound on the elements of $\mathrm{Spec}(\Delta_r)$ in terms of Ω, h_M, and r. Interestingly, the lower bound is both simpler and larger when the Yang–Mills equation is satisfied.

The relationship between $\mathrm{Spec}(\Delta_r)$ and the eigenvalues and eigenspaces of Δ, which we establish in Section 3, depends on a complete understanding of the Peter–Weyl theorem. It is not enough to know that the matrix entries of irreducible representations of G form a dense set in $L^2(G,\mathbb{C})$. We need to exhibit a $G \times G$-equivariant unitary equivalence between $L^2(G,\mathbb{C})$ (with $G \times G$ acting via pull-back by left and right multiplication) and the Hilbert space direct sum of the representations of $G \times G$ obtained by applying the "Hom" functor to the irreducible representations of G. There are books which do this (e.g., Adams, 1969; Wallach, 1973); however, a streamlined account of precisely what is needed seems preferable, because we need to establish much notation, which forms a major part of the Peter–Weyl theorem anyway. Also, we need to extend the Peter–Weyl theorem to compact homogeneous spaces, in order to fully understand the relationship between the holonomy group and representations that admit particles of zero mass. At any rate, Sections 1 and 2 provide a concise treatment of harmonic analysis on compact homogeneous spaces for those who need one, and an index of notation for those who do not.

In Section 6, there are a number of comments, problems, and physical speculations which are perhaps more interesting than correct or verifiable. In particular, we offer an explanation for why there seems to be a positive lower bound on the set of masses of all electrically charged particles. Also, we suggest how nature prefers to exhibit more particles arising from one representation of the gauge group than another representation. Implicit in these explanations is that certain properties of particles, such as their masses and relative tendencies to exist, can never be explained through local invariants or purely algebraic manipulations, but depend on the *global* geometry of the space-time-charge continuum. The same situation exists in differential geometry; the eigenvalues and eigenspaces of the Laplace operator are never determined by the metric on a small piece of the manifold, but are influenced by every piece. The properties of the smallest constituents of the universe may depend critically on the universe at large.

1. ALGEBRAIC PRELIMINARIES

We recall some basic facts about unitary representations, while establishing notation.

Let G be a group and let $r\colon G \to U(W_r)$ be a unitary representation, where W_r is a finite-dimensional complex vector space with Hermitian inner product $\langle\,,\rangle_r$ and $U(W_r) \equiv \{A \in \mathrm{Hom}(W_r, W_r) \mid \langle Au, Av\rangle_r = \langle u, v\rangle_r \forall u, v \in W_r\}$ where $\mathrm{Hom}(W_r, W_r)$ is the space of all linear $A\colon W_r \to W_r$. Given another such representation $s\colon G \to U(W_s)$, we say that a linear $A\colon W_r \to W_s$ is G-equivariant if $A \circ r(g) = s(g) \circ A, \forall g \in G$. If there is such an A which is an isomorphism, we write $r \sim s$; and if $\langle Au, Av\rangle_s = \langle u, v\rangle_r$ for all $u, v \in W_r$, A is called a unitary equivalence. We say r is irreducible if there are no invariant subspaces $V \subset W_r$ [i.e., $r(g)(V) \subset V$ for all $g \in G$] other than $V = 0$ and $V = W_r$.

> *Lemma 1.1* (Schur). If r and s are irreducible and $A\colon W_r \to W_s$ is G-equivariant, then either $A = 0$ or αA is a unitary equivalence for some $\alpha > 0$.

Proof. Let $B\colon W_r \to W_r$ be the unique linear map such that $\langle Av, Aw\rangle_s = \langle Bv, w\rangle_r \forall v, w \in W_r$. A simple computation shows that B is G-equivariant, and hence the eigenspaces of B are invariant. Since r is irreducible, there is only one eigenspace W_r, and $B = zI$ for some $z \in \mathbb{C}$. Actually, $z \in \mathbb{R}$ and $z \geqslant 0$, because $\langle Av, Av\rangle_s = z\langle v, v\rangle_r$. Since A is G-equivariant $A(W_r)$ is invariant, whence s irreducible yields $A = 0$ or A onto with $\sqrt{z}A$ a unitary equivalence. ■

If $r_1, \dots, r_n$ are unitary representations of G, then we can form the orthogonal direct sum $W_{r_1} \oplus \cdots \oplus W_{r_n}$ with Hermitian inner product induced by the $\langle\,,\rangle_{r_i}$ and obtain a unitary representation $r_1 \oplus \cdots \oplus r_n\colon G \to U(W_{r_1} \oplus \cdots \oplus W_{r_n})$.

> *Lemma 1.2.* Let $s, r_1, \dots, r_n$ be unitary representations of G with r_i irreducible and $r_i \not\sim r_j$ for $i \neq j$, $1 \leqslant i$, $j \leqslant n$. Suppose $F\colon W_{r_1} \oplus \cdots \oplus W_{r_n} \to W_s$ is G-equivariant, onto, and nonzero on each summand. Then there are positive constants $a_1, \dots, a_n$, such that F is a unitary equivalence if $\langle\,,\rangle_{r_i}$ is replaced by $a_i\langle\,,\rangle_{r_i}$.

Proof. Since the kernel of $F\colon W_{r_i} \to F(W_{r_i})$ is invariant and not W_{r_i}, we have that $F\colon W_{r_i} \to F(W_{r_i})$ is a G-equivariant isomorphism and by Schur's lemma there is a constant $\alpha_i > 0$ such that $\alpha_i F\colon W_{r_i} \to F(W_{r_i})$ is unitary, whence $F\colon W_{r_i} \to F(W_{r_i})$ is unitary if $\langle\,,\rangle_{r_i}$ is replaced by $a_i\langle\,,\rangle_{ri}$, $a_i \equiv \alpha_i^{-2}$. It remains to prove that $F(W_{r_i}) \perp F(W_{r_j})$ for $i \neq j$. Let $P_i\colon W_s \to F(W_{r_i})$ be the (necessarily G-equivariant) orthogonal projection onto $F(W_{r_i})$, and note that the restriction of P_i to $F(W_{r_j})$ is zero by Schur's lemma, since $r_i \not\sim r_j$. Thus, $F(W_{r_i}) \perp F(W_{r_j})$. ■

For unitary representations r and s, let $\mathrm{Hom}(W_r, W_s)$ ($\equiv$ the set of all linear maps from W_r to W_s) have the Hermitian inner product ${}_r\langle A, B\rangle_s = \mathrm{tr}(B^*A) = \Sigma_i\langle B^*A(e_i), e_i\rangle_r = \Sigma_i\langle A(e_i), B(e_i)\rangle_s$, where $\{e_i \mid i = 1, \dots, d_r\}$ is any frame for W_r and $B^* \in \mathrm{Hom}(W_s, W_r)$ is the adjoint of $B \in \mathrm{Hom}(W_r, W_s)$ (i.e., $\langle Bv, w\rangle_s = \langle v, B^*w\rangle_r \forall v \in W_r, w \in W_s$). For $w \in W_r$ and $v \in W_s$, let $v \otimes w \in \mathrm{Hom}(W_r, W_s)$ be given by $(v \otimes w)(v') = \langle v', w\rangle_r v, \forall v' \in W_r$. A simple computation shows ${}_r\langle v \otimes w, v' \otimes w'\rangle_s = \langle v, v'\rangle_s\langle w', w\rangle_r$. There is a representation $r \times s\colon G \times G \to U(\mathrm{Hom}(W_r, W_s))$, given by $(r \times s)(g_1, g_2)(A) = s(g_2) \circ A \circ r(g_1^{-1})$; a simple calculation shows $r \times s$ to be unitary.

> *Lemma 1.3.* Let r, s, r', s' be irreducible unitary representations of G. Then $r \times s$ and $r' \times s'$ are irreducible with $r \times s \sim r' \times s'$ only if $r \sim r'$ and $s \sim s'$.

Proof. Let $F\colon \mathrm{Hom}(W_r, W_s) \to \mathrm{Hom}(W_{r'}, W_{s'})$ be any $G \times G$-equivariant map. If $r \not\sim r'$ or $s \not\sim s'$, we will show $F = 0$, whence $r \times s \not\sim r' \times s'$. If $r = r'$ and $s = s'$, then we show that $F = aI$ for some $a \in \mathbb{Z}$, and so there can be no orthogonal projections onto invariant subspaces other than 0 or

$\mathrm{Hom}(W_r, W_s)$ (i.e., $r \times s$ is irreducible). Now for $w \in W_r$ and $w' \in W_{r'}$ (respectively, $v \in W_s, v' \in W_{s'}$) we have a unique linear map $K(w', w)$: $W_s \to W_{s'}$ (respectively, $L(v, v')$: $W_{r'} \to W_r$) such that $\langle K(w', w)v, v' \rangle_{s'} = {}_{s'}\langle F(v \otimes w), v' \otimes w' \rangle_{r'} = \langle L(v, v')(w'), w \rangle_r$ for all $v \in W_s, v' \in W_{s'}$ (respectively, for all $w \in W_r, w' \in W_{r'}$). Using the $G \times G$-equivariance of F, we have that $K(w', w)$ and $L(v, v')$ are G-equivariant. Thus, $K(w', w) = 0$ if $s \not\sim s'$ and $L(v, v') = 0$ if $r \not\sim r'$; and in either case $F = 0$. If $r = r'$ and $s = s'$, we have that $K(w', w)$ and $L(v, v')$ are scalar multiples of the identity, whence

$$\begin{aligned} {}_s\langle F(v \otimes w), v' \otimes w' \rangle_r &= a \langle v, v' \rangle_s \langle w, w' \rangle_r \\ &= {}_s\langle a(v \otimes w), v' \otimes w' \rangle_r, \quad \forall w, w' \in W_r \quad \text{and} \quad v, v' \in W_s \end{aligned}$$

Since $\{v \otimes w \mid v \in W_s, w \in W_r\}$ spans $\mathrm{Hom}(W_r, W_s)$, we have $F = aI$. ■

2. HARMONIC ANALYSIS ON COMPACT HOMOGENEOUS SPACES

We define the Laplace operator and related notions on a compact Lie group. Using the results of Section 1, we then establish a complete, equivariant version of the Peter–Weyl theorem and its extension to compact homogeneous spaces.

Let e be the identity of the compact, Lie groups G and identify the Lie algebra $\mathcal{G}$ with $T_eG \equiv$ tangent space of G at e. The $\mathscr{A}\mathfrak{d}$-invariant inner product k on $\mathcal{G}$ determines a bi-invariant Riemannian metric k_G on G; for $X, Y \in T_gG$, we set $k_G(X, Y) \equiv k(L_{g^{-1}*}X, L_{g^{-1}*}Y) = k(\mathscr{A}\mathfrak{d}_g L_{g^{-1}*}X, \mathscr{A}\mathfrak{d}_g L_{g^{-1}*}Y) = k(R_{g^{-1}*}X, R_{g^{-1}*}Y)$ where L_g (and R_g): $G \to G$ are left (and right) translation by g. Let $X_1, \ldots, X_m$ be an o.n. basis of T_eG and let $\bar{X}_1, \ldots, \bar{X}_m$ be the L_g-invariant extensions ($\bar{X}_{ig} \equiv L_{g*}X_i$). Since $g \mapsto g \cdot \exp(tX_j)$ is a one-parameter group of isometries generated by $\bar{X}_i$, we have that $\bar{X}_i$ is a Killing vector field. Since $\bar{X}_i$ has constant length as well, we know (Kabayashi and Nomizu, 1963, p. 252) that the integral curves $t \mapsto g \cdot \exp(tX_i)$ of $\bar{X}_i$ are geodesics. Letting D_t denote differentiation with respect to t, the Laplace operator for (G, k_G) is given (at $t = 0$) by $-\Delta_G(f)(g) = \Sigma_i D_t^2[f(g \exp tX_i)] = \Sigma_i \bar{X}_i^2[f]$ (i.e., $\Delta_G = -\Sigma_i \bar{X}_i^2$) where we view $\bar{X}_i$ as a differential operator on $C^\infty(G, \mathbb{C})$, the space of C^∞ complex-valued functions on G.

There is a Hermitian inner product on $C^\infty(G; \mathbb{C})$ given by $(u, v)_G = \int_G u\bar{v}\mu_G$ where μ_G is the volume element of (G, k_G), and note that $L^2(G; \mathbb{C})$ is the completion of $C^\infty(G; \mathbb{C})$ to a Hilbert space. Since the maps L_g and R_g are isometries, we have unitary representations L, R: $G \to U(L^2(G; \mathbb{C}))$ given by $L(g)f = f \circ L_{g^{-1}}$ and $R(g)f = f \circ R_g$. Since $[R(g), L(g)] = 0$, we have a unitary representation $L\times R$: $G\times G\to U(L^2(G;\mathbb{C}))$ given by $(L\times R)(g_1,g_2)=L(g_1)\circ R(g_2)$. We will show that there is a one-to-one correspondence between the $L\times R$ irreducible subspaces of $L^2(G;\mathbb{C})$ and the representations $r\times r$ of Section 1 as r ranges over a complete set of mutually inequivalent unitary representations of G. This is part of the Peter–Weyl theorem.

Let μ be an eigenvalue of Δ_G and $V_G(\mu)$ the eigenspace. From the general theory of the Laplace operator on arbitrary compact Riemannian manifolds (see Warner, 1971), we know that $\dim[V_G(\mu)]<\infty$ and the set of eigenvalues is a discrete set of nonnegative real numbers. Moreover, $L^2(G;\mathbb{C})=\oplus_\mu V_G(\mu)$, as an orthogonal Hilbert space direct sum. Since R_g and L_g are isometries (e.g., sending geodesics to geodesics), we have $[L(g),\Delta_G]=[R(g),\Delta_G]=0$, whence $V_G(\mu)$ is a $L\times R$-invariant subspace of $L^2(G;\mathbb{C})$.

To identify the irreducible subspaces of $V_G(\mu)$, we introduce Casimir operators. Given a unitary representation r: $G\to U(W_r)$, let r': $\mathcal{G}\to \operatorname{Hom}(W_r,W_r)$ be the representation of $\mathcal{G}$ given by $r'(A)(v)=D_t r(\exp tA)(v)$ at $t=0$. The Casimir operator C_r: $W_r\to W_r$ is given by $C_r=-\Sigma_i r'(X_i)\circ r'(X_i)$, which is independent of the choice of o.n. basis $X_1,\ldots,X_n$ of $\mathcal{G}$. Since r is unitary, $r'(A)$ is skew-adjoint relative to $\langle\,,\rangle_r$, and then we know that C_r is a nonnegative self-adjoint operator because $\langle C_r(v),w\rangle_r=\Sigma_i\langle r'(X_i)(v),r'(X_i)(w)\rangle_r$. Since

$$\begin{aligned} r(g)C_r r(g)^{-1} &= -\Sigma_i r(g)\circ r'(X_i)\circ r(g)^{-1}\circ r(g)\circ r'(X_i)\circ r(g)^{-1}\\ &= -\Sigma_i r'(\mathrm{Ad}_g X_i)\circ r'(\mathrm{Ad}_g X_i)=C_r \end{aligned}$$

we see that C_r is G-equivariant and hence W_r decomposes into an orthogonal direct sum of invariant eigenspaces of C_r. Also note that $r'(A)\circ r'(A)=D_t^2 r(\exp tA)$ at $t=0$, whence we could also define $C_r(v)$ as the Laplacian at e of the vector-valued function $g\mapsto r(g)(v)$ on G.

Let $\hat{G}$ be a complete set of mutually inequivalent irreducible unitary representations of G. For $r\in\hat{G}$, we have from the above that $C_r=c_r I$ for some constant $c_r\geqslant 0$; indeed, $c_r>0$ if $r'\neq 0$. For any $r\in\hat{G}$, we define a linear map Ψ_r: $\operatorname{Hom}(W_r,W_r)\to C^\infty(G,\mathbb{C})$ by $[\Psi_r(A)](g)=\operatorname{tr}[r(g)\circ A]={}_r\langle A,r(g^{-1})\rangle_r$ for $A\in\operatorname{Hom}(W_r,W_r)$ and $g\in G$. Recall that $\operatorname{Hom}(W_r,W_r)$ and $C^\infty(G,\mathbb{C})$ are both $G\times G$ representation spaces. A simple computation shows Ψ_r to be $G\times G$-equivariant. Note that

$$\begin{aligned} \Delta_G[\Psi_r(A)](g) &= -\Sigma_i D_t^2\operatorname{tr}[r(g\exp tX_i)A]\\ &= \operatorname{tr}[r(g)C_rA]=c_r\operatorname{tr}[r(g)A]=c_r\Psi_r(A)(g) \end{aligned}$$

and so $\Psi_r(\mathrm{Hom}(W_r, W_r)) \subset V_G(c_r)$. Let $\hat{G}(\mu)=\{r\in\hat{G}\,|\,c_r=\mu\}$. Since the representations $r\times r$ for $r\in\hat{G}(\mu)$ are mutually inequivalent irreducible representations by Lemma 1.3, it follows from Lemma 1.2 that the subspaces $\Psi_r(\mathrm{Hom}(W_r, W_r))$ for $r\in\hat{G}(\mu)$ are mutually orthogonal, whence $\hat{G}(\mu)$ is finite, since $\dim V(\mu)<\infty$. Let $\Psi_\mu=\oplus_{r\in\hat{G}(\mu)}\Psi_r$.

Theorem 2.1. The $G\times G$ equivariant map Ψ_μ: $\oplus_{r\in\hat{G}(\mu)}\mathrm{Hom}(W_r, W_r)\rightarrow V(\mu)$ is an isomorphism and replacing ${}_r\langle\,,\rangle_r$ by ${}_r\langle\,,\rangle_r'\equiv V(G)d_r^{-1}{}_r\langle\,,\rangle_r[d_r\equiv\dim(W_r), V(G)\equiv\int_G\mu_G]$, Ψ_μ becomes a unitary equivalence.

Proof. To prove Ψ_μ is an isomorphism, we need only prove Ψ_μ is onto because of Lemma 1.2. Let V be an arbitrary irreducible subspace of $V_G(\mu)$ relative to the representation R: $G\rightarrow U(V_G(\mu))$ defined earlier $[R(g)f=f\circ R_g]$. Since $V_G(\mu)$ is the direct sum of such subspaces, it suffices to prove image $(\Psi_\mu)\supset V$. There is some $r\in\hat{G}$ such that R: $G\rightarrow U(V)$ is unitarily equivalent to r via some E: $V\rightarrow W_r$. Let u be the unique function in V such that $(v,u)_G=v(e)$ for all $v\in V$; indeed, $u=\Sigma_i\overline{u_i(e)}u_i$ for any o.n. basis $\{u_i\}$ of V. For any $f\in V$, we have

$$\begin{aligned}\Psi_r(E(f)\otimes E(u))(g)&=\mathrm{tr}\{r(g)\circ[E(f)\otimes E(u)]\}\\&=\mathrm{tr}\{[r(g)(E(f))]\otimes E(u)\}\\&=\langle r(g)(E(f)), E(u)\rangle_r\\&=\langle E(f\circ R_g), E(u)\rangle_r=(f\circ R_g, u)_G\\&=(f\circ R_g)(e)=f(g)\end{aligned}$$

This proves $f\in$ image Ψ_r and $c_r=\mu$, whence $V\subset$ image Ψ_μ. In view of Lemma 1.2, we need only replace ${}_r\langle\,,\rangle_r$ by $a_{rr}\langle\,,\rangle_r$ to make Ψ_μ a unitary equivalence. To find a_r, let $e_1,\ldots,e_{d_r}$ be an o.n. basis of W_r. Then for any $w\in W_r$, we have

$$\begin{aligned}d_ra_r|w|_r^2&=\Sigma_ia_r\langle e_i,e_i\rangle_r\langle w,w\rangle_r\\&=\Sigma_ia_{rr}|e_i\otimes w|_r^2=\Sigma_i\|\Psi_r(e_i\otimes w)\|_G^2\\&=\int_G\Sigma_i|\langle r(g)e_i,w\rangle_r|^2\mu_G(g)=\int_G|w|_r^2\mu_G(g)=|w|_r^2V(G)\end{aligned}$$

whence

$$a_r=V(G)d_r^{-1}\qquad\blacksquare$$

Since Ψ_r is an isomorphism, we should be able to recover $A \in \operatorname{Hom}(W_r, W_r)$ from $\Psi_r(A) \in V_G(c_r)$.

Proposition 2.2. For any $A \in \operatorname{Hom}(W_r, W_r)$, we have

$$A = d_r V(G)^{-1} \int_G \Psi_r(A)(g) r(g^{-1}) \mu_G(g)$$

Proof.

$$\begin{aligned}
\Psi_r(A)(g') &= \operatorname{tr}[r(g')A] = {}_r\langle A, r(g')^* \rangle_r \\
&= d_r V(G)^{-1} (\Psi_r(A), \Psi_r(r(g')^*))_G \\
&= d_r V(G)^{-1} \int_G \Psi_r(A)(g)\, \overline{\operatorname{tr}[r(g) r(g')^*]}\, \mu_G(g) \\
&= d_r V(G)^{-1} \int_G \Psi_r(A)(g) \operatorname{tr}[r(g') r(g^{-1})] \mu_G(g) \\
&= \operatorname{tr}\left[r(g') d_r V(G)^{-1} \int_G \Psi_r(A)(g) r(g^{-1}) \mu_G(g) \right] \\
&= \Psi_r\left(d_r V(G)^{-1} \int_G \Psi_r(A)(g) r(g^{-1}) \mu_g(g) \right)(g') \qquad \blacksquare
\end{aligned}$$

Let H be a closed (necessarily Lie) subgroup of G. Then $G/H \equiv \{gH \mid g \in G\}$ has a unique C^∞ structure such that Q: $G \to G/H$ is a C^∞ principal bundle with group H acting on G to the right by isometries of (G, k_G). Let $k_{G/H}$ be the metric tensor on G/H induced by k_G via Q [i.e., $k_{G/H}(Q_* X, Q_* Y) = k_G(X, Y)$], and let $\Delta_{G/H}$ be the Laplace operator for $k_{G/H}$. One can prove that the horizontal lifts to G of geodesics on G/H are geodesics on G and the fibers (cosets of H) are totally geodesic. Consequently, $(\Delta_{G/H} u) \circ Q = \Delta_G(u \circ Q)$ for any $u \in C^\infty(G/H; \mathbb{C})$. The map Q: $G \to G/H$ is G-equivariant relative to left multiplication by G on G and G/H. We note that Q^*: $C^\infty(G/H; \mathbb{C}) \to C^\infty(G; \mathbb{C})$ is a G-equivariant isomorphism onto the space $C^\infty(G, H; \mathbb{C}) \equiv \{ f \in C^\infty(G, \mathbb{C}) \mid f \circ L_h = f \text{ for all } h \in H\}$, where $Q^*(f) \equiv f \circ Q$. Since $Q^* \circ \Delta_{G/H} = \Delta_G \circ Q^*$, for each eigenvalue μ of $\Delta_{G/H}$ with eigenspace $V_{G/H}(\mu)$, we have $Q^*(V_{G/H}(\mu)) = V_G(\mu) \cap C^\infty(G, H; C)$. If $\mu_{G/H}$ is the volume element of $k_{G/H}$, then we have the inner product on $C^\infty(G/H; \mathbb{C})$ given by $(u, v)_{G/H} \equiv \int_{G/H} u \bar{v} \mu_{G/H}$. Note that $(u, v)_{G/H} = V(H)^{-1} (Q^* u, Q^* v)_G$.

For $r \in \hat{G}$, we define $H_r = \{ w \in W_r \mid r(h)(w) = w, \forall h \in H\}$. Let $\operatorname{Hom}(W_r, H_r)$ be the $G \times \{e\}$-invariant subspace of all $A \in \operatorname{Hom}(W_r, W_r)$ such that $A(W_r) \subset H_r$, or equivalently, $r(h) \circ A = A$ for all $h \in H$. For

$A \in \mathrm{Hom}(W_r, H_r)$, we have

$$\Psi_r(A)(gh) = \mathrm{tr}[r(gh)A] = \mathrm{tr}[r(g)r(h)A] = \Psi_r(A)(g)$$

whence $\Psi_r(A) \in V_G(c_r) \cap C^\infty(G, H; \mathbb{C})$. Conversely, if $\Psi_r(A) \in C^\infty(G, H; \mathbb{C})$, then using Proposition 2.2, we obtain

$$\begin{aligned} r(h) \circ A &= d_r V(G)^{-1} \int_G \Psi_r(A)(g) r(hg^{-1}) \mu_G(g) \\ &= d_r V(G)^{-1} \int_G \Psi_r(A)(g'h) r(g'^{-1}) \mu_G(g') = A \qquad \text{for all } h \in H \end{aligned}$$

whence $A \in \mathrm{Hom}(W_r, H_r)$. Since the spaces $\Psi_r(\mathrm{Hom}(W_r, W_r))$ are $G \times \{e\}$-invariant (indeed, $G \times G$-invariant), we have

$$\begin{aligned} Q^*\big(V_{G/H}(\mu)\big) &= V_G(\mu) \cap C^\infty(G, H; \mathbb{C}) \\ &= \oplus_{r \in \hat{G}(\mu)} [\Psi_r(\mathrm{Hom}(W_r, W_r)) \cap C^\infty(G, H; \mathbb{C})] \\ &= \oplus_{r \in \hat{G}(\mu)} \Psi_r(\mathrm{Hom}(W_r, H_r)) \end{aligned}$$

Theorem 2.3. Giving $\mathrm{Hom}(W_r, H_r)$ the inner product ${}_r\langle , \rangle_r^H \equiv V(G/H) d_r^{-1} \, {}_r\langle , \rangle_r$ we have a unitary equivalence of G-representations [G acts as $G \times \{e\}$ on $\mathrm{Hom}(W_r, H_r)$]

$$Q^{*-1} \circ \Psi_\mu \colon \oplus_{r \in \hat{G}(\mu)} \mathrm{Hom}(W_r, H_r) \to V_{G/H}(\mu)$$

Proof. We have seen that $Q^*\colon V_{G/H}(\mu) \to V_G(\mu) \cap C^\infty(G, H; \mathbb{C})$ and $\Psi_\mu\colon \oplus_{r \in \hat{G}(\mu)} \mathrm{Hom}(W_r, W_r) \to V_G(\mu) \cap C^\infty(G, H; \mathbb{C})$ are both G-equivariant isomorphisms, and we know from before that Ψ_μ preserves the orthogonality of the summands. Hence we only need unitarity on each summand. We have

$$\begin{aligned} \|(Q^{-1*} \circ \Psi_r)(A)\|_{G/H}^2 &= V(H)^{-1} \|\Psi_r(A)\|_G^2 = V(H)^{-1} V(G) d_r^{-1} \, {}_r|A|_r^2 \\ &= V(G/H) d_r^{-1} \, {}_r|A|_r^2 = {}_r|A|_r^{H2} \end{aligned}$$ ■

Remark. We can form the Hilbert space direct sum of the spaces $\mathrm{Hom}(W_r, H_r)$ using the inner products ${}_r\langle , \rangle_r^H$. Then Theorem 2.3 immediately yields a unitary equivalence $\Psi\colon \oplus_{r \in \hat{G}} \mathrm{Hom}(W_r, H_r) \to L^2(G/H; \mathbb{C})$ of representations of G; indeed, representations of $G \times G$ when $H = \{e\}$. This is the Peter–Weyl theorem, but some readers may not recognize it as such. Let $e_1, \ldots, e_{d_r}$ be an o.n. basis of W_r such that $e_1, \ldots, e_{h_r}$ span H_r ($h_r \leqslant d_r$). Define $r_{ij} \in C^\infty(G; \mathbb{C})$ by $r(g)(e_j) = \Sigma r_{ij}(g) e_i$. Now $d_r V(G/H)^{-1} e_i \otimes e_j$ for $1 \leqslant i \leqslant h_r$ and $1 \leqslant j \leqslant d_r$ form an o.n. basis of $\mathrm{Hom}(W_r, H_r)$ and $\Psi_r(e_i \otimes e_j)(g) = \mathrm{tr}[r(g) e_i \otimes e_j] = \langle r(g) e_i, e_j \rangle_r = r_{ji}(g) \in$

$C^\infty(G, H; \mathbb{C})$. Thus, $\{d_r V(G/H)^{-1} Q^{*-1} r_{ji} | r \in \hat{G}, 1 \leqslant i \leqslant h_r, 1 \leqslant j \leqslant d_r\}$ is an o.n. basis of $L^2(G/H)$. Since any $f \in C^\infty(G/H)$ can be uniformly approximated by finite linear combinations of eigenfunctions of $\Delta_{G/H}$ (see Warner, 1971), f can be uniformly approximated by finite linear combinations of the r_{ji}.

3. HARMONIC ANALYSIS ON PRINCIPAL BUNDLES

Building upon the notation of the introduction, for $A \in \mathcal{G}$, let A^* be the vector field on P given by $A^*_p = D_t(p \exp tA)$ at $t = 0$. Let F_p: $G \to pG \equiv \{pg | g \in G\}$ be defined by $F_p(g) = pg$. For $A, B \in \mathcal{G}$ with left-invariant extensions $\bar{A}$, $\bar{B}$, note that $h(F_{p*g}\bar{A}, F_{p*g}\bar{B}) = h(D_t F_p(g \exp tA), D_t F_p(g \exp tB)) = h(A^*_{pg}, B^*_{pg}) = k(\omega(A^*_{pg}), \omega(B^*_{pg})) = k(A, B) = k_G(\bar{A}_g, \bar{B}_g)$, so that F_p is an isometry and $A^* = F_{p*}(\bar{A})$. Moreover, for an o.n. basis $X_1, \ldots, X_m$ of $\mathcal{G}$, $\Delta^V \equiv -(X_1^{*2} + \cdots + X_m^{*2})$: $C(P, \mathbb{C}) \to C^\infty(P, \mathbb{C})$ is an operator such that $(\Delta^V u)(p)$ is the Laplacian of $u | pG$ at p regarded as a function on pG with the metric $h | pG$. The vector field A^* generates the one-parameter group $p \mapsto p \exp tA$ of isometries of (P, h), whence A^* is a Killing vector field. Since A^* has constant length, we know (Kabayashi and Nomizu, 1963) that the integral curves $t \mapsto p \exp tX_i$ of A^* are geodesics in (P, h) as well as pG [i.e., pG is a totally geodesic submanifold of (P, h)].

Recall that Δ is the Laplacian of (P, h). Regarding A^* as a differential operator, we have $[\Delta, A^*] = 0$ since A^* is Killing; so $[\Delta^V, \Delta] = 0$. Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots$ be the eigenvalues of Δ with corresponding eigenspaces $V(\lambda_i)$ $i = 0, 1, 2, \ldots$. Since $[\Delta, \Delta^V] = 0$, we have $\Delta^V(V(\lambda_i)) \subset V(\lambda_i)$. Moreover, since A^* is Killing and hence divergence free, it is a skew-adjoint linear operator on $C^\infty(P, \mathbb{C})$ with the inner product $(u, v) = \int_P u\bar{v}\mu_h$ [i.e., $(A^*[u], v) = -(u, A^*[v])$]. Thus, $\Delta^V = -(X_1^{*2} + \cdots + X_m^{*2})$ is symmetric and nonnegative, and $V(\lambda_i)$ decomposes into a direct sum of orthogonal subspaces

$$V(\lambda_i) = \oplus_{\mu \geqslant 0} V(\lambda_i) \cap V^V(\mu)$$

where $V^V(\mu) = \{u \in C^\infty(P, \mathbb{C}) | \Delta^V u = \mu u\}$, and all but finitely many summands are 0. Note that since $u \in V^V(\mu)$ implies $u | pG$ is an eigenfunction on $(pG, h | pG) \cong (G, k_G)$ with eigenvalue μ, we have that $V^V(\mu) = 0$, unless μ is an eigenvalue of Δ_G; we may assume the above sum is over such μ.

For any representation r: $G \to GL(V_r)$ we set $C(P, V_r) = \{f$: $P \to V_r | f(pg) = r(g^{-1})[f(p)] \forall g \in G$; f is $C^\infty\}$. Note that $C(P, V_r)$ can be identified with the space of sections of the associated vector bundle $P \times_G V_r \to M$. If r is unitary, then we have an inner product on $C(P, V_r)$ given by $(f_1, f_2)_r =$

$V(G)^{-1}\int_P \langle f_1, f_2\rangle_r \mu_h$. Let $\mathrm{Hom}(\mu)$ (with inner product $\langle\,,\rangle_\mu$) be the orthogonal sum $\oplus_{r\in\hat{G}(\mu)}\mathrm{Hom}(W_r, W_r)$ and let r_μ: $G\times G\to U(\mathrm{Hom}(\mu))$ be the direct sum of the representations $r\times r$, $r\in\hat{G}(\mu)$, where $\mathrm{Hom}(W_r, W_r)$ has the inner product ${}_r\langle\,,\rangle_r'$ making Ψ_μ: $\mathrm{Hom}(\mu)\to V_G(\mu)$ a unitary equivalence. Relative to the unitary representation $G\to\{e\}\times G\to U(\mathrm{Hom}(W_r, W_r))$, we have the space $C(P,\mathrm{Hom}(W_r, W_r)_R)$ consisting of all f: $P\to\mathrm{Hom}(W_r, W_r)$ such that $f(pg)=r(g^{-1})\circ f(p)$. We have a unitary representation R_r: $G\to U(C(P,\mathrm{Hom}(W_r, W_r)_R))$ given by $[R_r(g)f](p)=f(p)\circ r(g^{-1})$. These piece together to give a unitary representation R_μ: $G\to U(C(P,\mathrm{Hom}(\mu)_R))$. We also have a unitary representation R: $G\to U(L^2(P,\mathbb{C}))$ given by $R(g)f=f\circ R_g=R_g^*f$. Since $[\Delta^V, R_g^*]=0$, $V^V(\mu)$ is an invariant subspace.

Lemma 3.1. The linear map $\mathfrak{F}_\mu$: $C(P,\mathrm{Hom}(\mu)_R)\to V^V(\mu)$ given by $\mathfrak{F}_\mu(f)(p)=\Psi_\mu(f(p))(e)$ is a unitary equivalence of R_μ with R: $G\to U(V^V(\mu))$.

Proof. We prove $\mathfrak{F}_\mu(f)\in V^V(\mu)$ by showing that $\mathfrak{F}_\mu(f)$ restricted to an arbitrary fiber pG is an eigenfunction of Δ_G when it is pulled back by F_p: $G\to pG$. Indeed, $[\mathfrak{F}_\mu(f)\circ F_p](g)=\mathfrak{F}_\mu(f)(pg)=\Psi_\mu(f(pg))(e)=\Psi_\mu(r_\mu(e,g^{-1})f(p))(e)=\Psi_\mu(f(p))(g^{-1})=[\Psi_\mu(f(p))\circ\mathrm{Inv}](g)$ where Inv: $G\to G$ is the isometry $g\mapsto g^{-1}$ of (G, k_G). Since $\Psi_\mu(f(p))\in V_G(\mu)$, we then have $\mathfrak{F}_\mu(f)\circ F_p\in V_G(\mu)$, and so $\mathfrak{F}_\mu(f)\in V^V(\mu)$. Moreover,

$$\int_{pG}|\mathfrak{F}_\mu(f)|^2\mu_{pG}=\int_G|\mathfrak{F}_\mu(f)\circ F_p|^2\mu_G=\int_G\left|\Psi_\mu(f(p))\circ\mathrm{Inv}\right|^2\mu_G$$

$$=\int_G\left|\Psi_\mu(f(p))\right|^2\mu_G=|f(p)|_\mu^2=V(G)^{-1}\int_{pG}|f|_\mu^2\mu_{pG}$$

whence

$$\int_P|\mathfrak{F}_\mu(f)|^2\mu_h=V(G)^{-1}\int_P|f|_\mu^2\mu_h=(f,f)_\mu$$

Thus, $\mathfrak{F}_\mu$ is an isometry onto its image. It is straightforward to check that the inverse of $\mathfrak{F}_\mu$ is given by $\mathfrak{F}_\mu^{-1}(u)(p)=\Psi_\mu^{-1}(u\circ F_p\circ\mathrm{Inv})$. The equivariance of $\mathfrak{F}_\mu$ follows from a simple calculation, using the $G\times G$ equivariance of Ψ_μ. ■

Using Proposition 2.2 and writing $\mathfrak{F}_\mu^{-1}(u)=\oplus_{r\in\hat{G}(\mu)}f_r$ where $u\in V^V(\mu)$ and $f_r\in C(P,\mathrm{Hom}(W_r, W_r)_R)$, we have $f_r(p)=d_rV(G)^{-1}\int_G u(pg^{-1})r(g^{-1})\mu_G(g)=d_rV(G)^{-1}\int_G u(pg)r(g)\mu_G(g)$, since Inv is an isometry of (G, k_G). Also, observe that $\mathfrak{F}=\oplus_\mu\mathfrak{F}_\mu$: $\oplus_{r\in\hat{G}}C(P,\mathrm{Hom}(W_r, W_r)_R)\to L^2(P,\mathbb{C})$ is a unitary equivalence. Nearly all particle fields of interest can be faithfully encoded into the space $C^\infty(P,\mathbb{C})$. Indeed, choosing any basis $e_1,\ldots,e_{d_r}$ of W_r, we have a decomposition of $\mathrm{Hom}(W_r,W_r)$ into $\{e\}\times G$-invariant subspaces $W_r(i)=\{w\otimes e_i \mid w\in W_r\}$. Thus, $C(P,\mathrm{Hom}(W_r,W_r)_R)\cong \oplus_i C(P,W_r(i))\cong d_r C(P,W_r)$; and so any particle field in $C(P,V_s)$, where V_s contains at most d_r copies of W_r, can be represented by a function in $C^\infty(P,\mathbb{C})$ via $\mathfrak{F}$. Of course, taking $G=G_1\times \mathrm{Spin}(n)$, where G_1 is the internal symmetry group, we can represent spinor fields within $C^\infty(P,\mathbb{C})$ too! We can obtain a basis-free decomposition of $\oplus_{r\in\hat{G}}C(P,\mathrm{Hom}(W_r,W_r)_R)$ into finite-dimensional invariant subspaces by pulling back via $\mathfrak{F}$ the decomposition $L^2(P,\mathbb{C})=\oplus_i V(\lambda_i)$ and taking intersections. Setting $C(r,\lambda_i)=\mathfrak{F}^{-1}(V(\lambda_i))\cap C(P,\mathrm{Hom}(W_r,W_r)_R)$, we will find $\oplus_{r\in\hat{G}}C(P,\mathrm{Hom}(W_r,W_r)_R)=\oplus_{i,r}C(r,\lambda_i)$ and $L^2(P,\mathbb{C})=\oplus_{i,r}\mathfrak{F}(C(r,\lambda_i))$. It is not difficult to show that every irreducible subrepresentation of R_r: $G\to U(C(r,\lambda_i))$ is equivalent to r. We will show that $C(r,\lambda_i)$ is closely related to the space of all particle fields, coming from the representation r, which have mass^2 $\lambda_i-c_r\geqslant 0$.

Every $X\in T_pP$ has a unique decomposition into horizontal and vertical vectors $X=X^H+X^V$, $X^H\in H_p\equiv\{X\in T_pP\mid\omega(X)=0\}$ and $X^V\in V_p\equiv\{X\in T_pP\mid\pi_*(X)=0\}$. For any q-form φ on P, we define φ^H by $\varphi^H(X_1,\ldots,X_q)=\varphi(X_1^H,\ldots,X_q^H)$. When $q=1$, we have $\varphi=\varphi^H+\varphi^V$ where $\varphi^V(X)=\varphi(X^V)$. For a unitary representation s: $G\to U(V_s)$, we have the space $\Lambda^q(P,V_s)$ of all V_s-valued q-forms on P. There is a natural inner product on $\Lambda^q(P,V_s)$ given by ${}_h(\varphi,\varphi')_s=V(G)^{-1}\int_P {}_h\langle\varphi,\varphi'\rangle_s\mu_h$, where ${}_h\langle\varphi,\varphi'\rangle_s(p)$ is the inner product of the V_s-valued forms on the vector space T_pP with metric h; see Bleecker (1981) for further details on this and what follows. We write ${}_h\|\varphi\|_s^2={}_h(\varphi,\varphi)_s$ (omitting h and s, if clear) and ${}_h|\varphi|_s^2={}_h\langle\varphi,\varphi\rangle_s$. For $\varphi\in\Lambda^1(P,V_s)$, we have $|\varphi|^2\equiv{}_h|\varphi|_s^2={}_h|\varphi^H|_s^2+{}_h|\varphi^V|_s^2$. If $\varphi\in\Lambda^q(P,V_s)$, we define $D\varphi\in\Lambda^{q+1}(P,V_s)$ by $D\varphi=(d\varphi)^H$.

There is also the space $\overline{\Lambda}^q(P,V_s)=\{\varphi\in\Lambda^q(P,V_s)\mid\varphi=\varphi^H$ and $R_g^*\varphi=s(g^{-1})\varphi\}$. One can check that $\overline{\Lambda}^q(P,V_s)$ is isomorphic to the space of all q-forms on M with values in the associated bundle $P\times_G V_s\to M$. Also $D(\overline{\Lambda}^q(P,V_s))\subset\overline{\Lambda}^{q+1}(P,V_s)$ and D corresponds to covariant differentiation of such forms. For clarity, we often write D as D_s.

There is an operator δ_s: $\overline{\Lambda}^{q+1}(P,V_s)\to\overline{\Lambda}^q(P,V_s)$ dual to D_s in the sense ${}_h(D_s\varphi,\psi)_s={}_h(\varphi,\delta_s\psi)_s$ for all $\varphi\in\overline{\Lambda}^q(P,V_s)$ and $\psi\in\overline{\Lambda}^{q+1}(P,V_s)$. We have an equivalent definition of δ_s, as follows. Let $\bar{*}$: $\overline{\Lambda}^q(P,V_s)\to\overline{\Lambda}^{n-q}(P,V_s)$ ($n=\dim M$) be defined by taking $\bar{*}(\varphi)$ to be the unique element of $\overline{\Lambda}^{n-q}(P,V_s)$ such that $(\bar{*}\varphi)|H_p=*_p(\varphi|H_p)$ where $*_p$ is the Hodge star for forms on H_p with the metric and volume element induced by h and $\pi^*(\mu_M)$. The self-adjoint operator $\Delta_s\equiv\delta_sD_s+D_s\delta_s$: $\overline{\Lambda}^q(P,V_s)\to\overline{\Lambda}^q(P,V_s)$ is the (Hodge) Laplacian. In the case $q=0$, we have $\overline{\Lambda}^0(P,V_s)=C(P,V_s)$ and $\Delta_s=\delta_sD_s$, since $\delta_s=0$ on $\overline{\Lambda}^0(P,V_s)$. In Euclidean field theory, $C(P,V_s)$ is the space of particle fields associated with s, and the eigenvalues of Δ_s:

$C(P,V_s)\to C(P,V_s)$ constitute the mass^2 spectrum $\mathrm{Spec}(\Delta_s)$ for such particle fields.

For any $u\in C^\infty(P,\mathbb{C})$, we define $\Delta^H u\in C^\infty(P,\mathbb{C})$ by taking $(\Delta^H u)(p)$ to be minus the sum of the second derivatives of f along a set of geodesics passing through p such that the tangent vectors at p form an o.n. basis of H_p. Since $H_p \perp V_p$ and pG is totally geodesic, it follows that the Laplace operator Δ of (P,h) is $\Delta^H+\Delta^V$, where Δ^V was introduced earlier. The operators Δ^H, Δ^V, and Δ extend to vector-valued functions on P. They each leave $C(P,V_s)$ invariant and commute with $\mathfrak{F}_\mu$. In particular, we have $\mathfrak{F}^{-1}(V(\lambda_i))=\oplus_r C(r,\lambda_i)$, and $C(P,\mathrm{Hom}(W_r,W_r)_R)=\oplus_i C(r,\lambda_i)$.

> *Lemma 3.2.* For any unitary representation s: $G\to U(V_s)$, the operators Δ_s and Δ^H on $C(P,V_s)$ are equal.

Proof. Recall (Warner, 1971) that $\Delta=d\delta$ where δ is the codifferential adjoint to d. For arbitrary $\varphi,\psi\in C(P,V_s)$, it suffices to prove $(\Delta_s\psi,\varphi)_s=(\Delta^H\psi,\varphi)_s$, but $(\Delta^H\psi,\varphi)_s=(\Delta\psi-\Delta^V\psi,\varphi)_s=(\delta d\psi,\varphi)_s-(\Delta^V\psi,\varphi)_s={}_h(d\psi,d\varphi)_s-{}_h(d\psi^V,d\varphi^V)_s={}_h(d\psi^H,d\varphi^H)_s={}_h(D_s\psi,D_s\varphi)_s=(\delta_s D_s\psi,\varphi)_s=(\Delta_s\psi,\varphi)_s$. ■

Recall that $C(r,\lambda_i)=\mathfrak{F}^{-1}(V(\lambda_i))\cap C(P,\mathrm{Hom}(W_r,W_r)_R)$, and $\mathrm{Spec}(\Delta_r)=\{\mu\in\mathbb{R}\mid\Delta_r f=\mu f$ for some $f\in C(P,W_r)$, $f\neq 0\}$. We define $C(P,W_r;m)$ to be the eigenspace of Δ_r with eigenvalue m.

> *Theorem 3.3.* For an irreducible unitary representation r: $G\to U(W_r)$, with Casimir operator $c_r I$, we have $\mathrm{Spec}(\Delta_r)=\{\lambda_i-c_r\mid C(r,\lambda_i)\neq 0\}$. Indeed, the (basis-dependent) G-equivariant isomorphism $C(P,\mathrm{Hom}(W_r,W_r)_R)\to d_r C(P,W_r)$ carries $C(r,\lambda_i)$ onto $d_r C(P,W_r;\lambda_i-c_r)$, the direct sum of d_r copies of $C(P,W_r;\lambda_i-c_r)$.

Proof. One can check that Δ, Δ^H, and Δ^V commute with the maps $\mathfrak{F}$: $C(P,\mathrm{Hom}(W_r,W_r)_R)\to V^V(c_r)$, and $C(P,\mathrm{Hom}(W_r,W_r)_R)\to d_r C(P,W_r)$. For $u\in V^V(c_r)\cap V(\lambda_i)$, we have $\Delta u=\lambda_i u$, $\Delta^V u=c_r u$, and $\Delta^H u=(\lambda_i-c_r)u$. Thus, the same holds for the corresponding $f=\Sigma_j f_j\otimes e_j\in C(r,\lambda_i)$ and its "components" f_j, $1\leqslant j\leqslant d_r$. In particular, $\Delta_r f_j=\Delta^H f_j=(\lambda_i-c_r)f_j$ [i.e., $f_j\in C(P,W_r;\lambda_i-c_r)$]. Conversely, if $f_j\in C(P,W_r;\lambda_i-c_r)$, then $f=\Sigma_j f_j\otimes e_j$ satisfies $\Delta^H f=(\lambda_i-c_r)f$. We already know $\Delta^V f=c_r f$ for $f\in C(P,\mathrm{Hom}(W_r,W_r)_R)$, since $\mathfrak{F}(f)\in V^V(c_r)$, and so $\Delta f=\lambda_i f$ and $f\in C(r,\lambda_i)$, as required. ■

> *Corollary3.4.* For $f\in C(P,W_r;\lambda_i-c_r)$, we have $\Delta f=\lambda_i f$, $\Delta^V f=c_r f$, and $\Delta_r f=\Delta^H f=(\lambda_i-c_r)f$. Moreover, $\|df\|^2=\lambda_i\|f\|^2$, $\|df^V\|^2=c_r\|f\|^2$, and $\|D_r f\|^2=\|(df)^H\|^2=(\lambda_i-c_r)\|f\|^2$. In particular, $\lambda_i-c_r\geqslant 0$.

Proof. The first statement follows from the proof of Theorem 3.3. Note that ${}_h\|D_rf\|_r^2 = {}_h(D_rf, D_rf)_r = (\delta_rD_rf, f)_r = (\Delta_rf, f)_r = (\lambda_i - c_r)\|f\|_r^2$, and the others follow similarly. ■

Recall that the field strength (or curvature) of the gauge potential (or connection) ω is $\Omega = D\omega \in \overline{\Lambda}^2(P, \mathcal{G})$ where the representation is $\mathfrak{Ad}\colon G \to GL(\mathcal{G})$.

Lemma 3.5. For any representation $r\colon G \to GL(V)$ and $f \in C(P, V)$, we have $Df = df + (r' \circ \omega)f \in \overline{\Lambda}^1(P, V)$ [i.e., $Df_p(X) = df_p(X) + r'(\omega(X))(f(p))$ for $p \in P$, $X \in T_pP$; and $r'\colon \mathcal{G} \to \operatorname{Hom}(V, V)$ is the corresponding Lie algebra representation]. Moreover, $D(Df) = (r' \circ \Omega)f \in \overline{\Lambda}^2(P, V)$.

Proof. We have $Df = df - df^V$, but $df^V(A_p^*) = df(A_p^*) = D_t f(p \exp tA) = D_t r(\exp(-tA))(f(p)) = -r'(A)(f(p)) = -r'(\omega(A^*))(f(p))$, and it follows that $Df = df + (r' \circ \omega)f$. Also, $d(Df) = d^2f + (r' \circ d\omega)f + (r' \circ \omega) \wedge df$. Since $d^2f = 0$, $\omega^H = 0$, and $(d\omega)^H = \Omega$, we obtain $D(Df) = (r' \circ \Omega)f$. ■

4. CONSTRAINTS IMPOSED BY PARTICLES OF ZERO MASS

Let $p \in P$ and let P_0 be the set of all points of P that can be joined to p by a smooth curve whose tangent vectors are all horizontal relative to ω. In Kabayashi and Nomizu (1963) it is proved that P_0 is a C^∞ immersed submanifold of P, and $\pi\colon P_0 \to M$ is a principal bundle with group $G_0 = \{g \in G \mid pg \in P_0\}$, called the holonomy group at p; $\pi\colon P_0 \to M$ is the holonomy bundle through p. The field strength Ω of ω at any $p_0 \in P_0$ has values in the Lie algebra $\mathcal{G}_0$ of G_0. Hence, the smaller G_0 is, the more "degenerate" Ω is. Indeed, if G_0 is finite, then $\Omega = 0$. If $G_0 = \{e\}$, then $\pi\colon P \to M$ is trivial as well, since P_0 is then the image of a global section of $\pi\colon P \to M$. Let G_0' be the closure of G_0; G_0' is a Lie subgroup of G. Let $\operatorname{Spec}(G/G_0')$ be the spectrum of the Laplace operator on G/G_0', as in Section 2.

Theorem 4.1. For each $\mu \in \operatorname{Spec}(G/G_0')$, there is at least one $r \in \hat{G}(\mu)$ such that $0 \in \operatorname{Spec}(\Delta_r)$.

Proof. Setting $H = G_0'$ in Theorem 2.3, we see that there is $r \in \hat{G}(\mu)$ with $\operatorname{Hom}(W_r, (G_0')_r) \neq 0$. Let $v \in (G_0')_r$, $v \neq 0$, and define $f \in C(P, W_r)$ by $f(p_0g) = r(g^{-1})(v)$ for any $p_0 \in P_0$ and $g \in G$. If $p_0g = p_0'g'$, then $p_0' = p_0gg'^{-1}$ whence $gg'^{-1} \in G_0 \subset G_0'$ and $r(g^{-1})(v) = r(g^{-1})[r(gg'^{-1})(v)] = r(g'^{-1})(v)$, and so f is well defined. Since $T_pP_0 \supset H_p$ and f is constant on P_0, we have $D_rf = (df)^H = 0$, and so $\Delta_rf = \delta_rD_rf = 0$. ■

Theorem 4.2. If $0 \in \operatorname{Spec}(\Delta_r)$ for some $r \in \hat{G}$, then $c_r \in \operatorname{Spec}(G/G_0')$. With more precision, $\dim[C(P, W_r; 0)] = \dim[(G_0')_r]$, where $(G_0')_r = \{w \in W_r \,|\, r(g)(w) = w \text{ for all } g \in G_0'\}$.

Proof. Let $f \in C(P, W_r; 0)$, and note that ${}_h(D_r f, D_r f)_r = (\Delta_r f, f)_r = 0$, whence $D_r f = 0$. Thus, f is constant on all horizontal curves, and so f is constant on P_0. For any $g \in G_0$ we have $f(p) = f(pg) = r(g^{-1})f(p)$, and by continuity this also holds for $g \in G_0'$; so $f(p) \in (G_0')_r$. The map $C(P, W_r; 0) \to W_r$ given by $f \mapsto f(p)$ is injective, and by the proof of 4.1, its image is $(G_0')_r$, whence $\dim[C(P, W_r; 0)] = \dim[(G_0')_r]$. Then we see that $0 \in \operatorname{Spec}(\Delta_r)$ implies $(G_0')_r \neq 0$, and Theorem 2.3 yields $c_r \in \operatorname{Spec}(G/G_0')$. ■

Remark. If G is connected, then r' determines r. We have seen that $c_r > 0$ when $r' \neq 0$. Hence, assuming G is connected, $c_r > 0$ if r is nontrivial. Thus, $0 \in \operatorname{Spec}(\Delta_r)$ for some nontrivial $r \in \hat{G}$ implies $0 < c_r \in \operatorname{Spec}(G/G_0')$, whence $\dim(G/G_0') \geqslant 1$, and $\operatorname{Spec}(G/G_0')$ is infinite. Then Theorem 4.1 gives us an infinite collection of $s \in \hat{G}$ for which $0 \in \operatorname{Spec}(\Delta_s)$.

5. LOWER BOUNDS ON MASS SPECTRA

For an arbitrary $r \in \hat{G}$, we find a lower bound on $\operatorname{Spec}(\Delta_r)$ in terms of the field strength (or curvature) $\Omega \in \Lambda^2(P, \mathscr{G})$, h_M, and r. The lower bound involves three constants, which we now define.

For each $p \in P$, we have a linear map $\Omega_r(p)$: $W_r \to \overline{\Lambda}^2(P, W_r)_p$ ($\equiv$ the space of W_r-valued 2-forms φ on T_pP such that $\varphi^H = \varphi$) given by $[\Omega_r(p)(v)](X, Y) = r'(\Omega(X, Y))(v)$ where r': $\mathscr{G} \to \operatorname{Hom}(W_r, W_r)$ comes from r: $G \to U(W_r)$. We define $b_r(p) = \max\{b \geqslant 0 \,|\, {}_h|\Omega_r(p)(v)|_r \geqslant b|v|_r \text{ for all } v \in W_r\}$, and $b_r = \min\{b_r(p) \,|\, p \in P\}$. We say that Ω is r-nondegenerate if $b_r > 0$; our lower bound on $\operatorname{Spec}(\Delta_r)$ is positive only when $b_r > 0$.

At each $p \in P$, we have another linear map $(\delta\Omega)_r(p)$: $W_r \to \overline{\Lambda}^1(P, W_r)$ defined by $[(\delta\Omega)_r(p)(v)](X) = r'(\delta\Omega(X))(v)$ where δ is the covariant codifferential dual to D: $\overline{\Lambda}^1(P, \mathscr{G}) \to \overline{\Lambda}^2(P, \mathscr{G})$. Define $Y_r(p) = \min\{b \geqslant 0 \,|\, {}_h|(\delta\Omega)_r(p)(v)|_r \leqslant b|v|_r \text{ for all } v \in W_r\}$, and $Y_r = \max\{Y_r(p) \,|\, p \in P\}$. Note that $Y_r = 0$ iff the Yang–Mills equation $\delta\Omega = 0$ holds.

The third constant is obtained by considering the map $\Omega_r^1(p)$: $\overline{\Lambda}^1(P, W_r) \to \overline{\Lambda}^1(P, W_r)$ defined by $[\Omega_r^1(p)(\sigma)](X) = \Sigma_i r'(\Omega(X, e_i))(v_i)$ where $e_1, \ldots, e_n$ is an o.n. basis of H_p and $\sigma = \Sigma v_i \otimes e_i^{\#}$, $v_i \in W_r$ [i.e., $\sigma(X) = \Sigma_i v_i h(X, e_i)$]. A simple computation shows that $\Omega_r^1(p)$ is independent of the choice of o.n. basis. We define $B_r(p) = \min\{b \geqslant 0 \,|\, {}_h|\Omega_r^1(p)(\sigma)|_r \leqslant b\,{}_h|\sigma|_r \text{ for all } \sigma \in \Lambda^1(P, W_r)_p\}$, and set $B_r = \max\{B_r(p) \,|\, p \in P\}$.

Lemma 5.1. If $f \in C(P, W_r)$, then $(r' \circ \Omega)f \in \Lambda^2(P, W_r)$, and

$$\delta_r[(r' \circ \Omega)f]_p = (\delta\Omega)_r(p)(f(p)) + \Omega_r^1(p)(D_r f_p)$$

Proof. Let $e_1, \ldots, e_n$ be an o.n. basis of H_p. We may extend $\pi_*(e_1), \ldots, \pi_*(e_n)$ to an o.n. frame field [defined on a neighborhood of $\pi(p)$] say $E'_1, \ldots, E'_n$ such that $[E'_i, E'_j] = 0$ at $\pi(p)$. Let $E_1, \ldots, E_n$ be the horizontal lifts of $E'_1, \ldots, E'_n$. Note that $\pi_*([E_i, E_j]) = [E'_i, E'_j]$, whence $[E_i, E_j]^H_p = 0$. A computation reveals that at p, we have

$$\begin{aligned}\delta_r[(r' \circ \Omega)f](E_j) &= -\Sigma_i E_i\big[r'\big(\Omega(E_i, E_j)\big)(f)\big]\\ &= -r'\big(\Sigma_i E_i\big[\Omega(E_i, E_j)\big]\big)(f) - \Sigma_i r'\big(\Omega(E_i, E_j)\big)(E_i[f])\\ &= r'\big(\delta\Omega(E_j)\big)(f) + \Sigma_i r'\big(\Omega(E_j, E_i)\big)(E_i[f])\\ &= [(\delta\Omega)_r(p)(f(p))](E_j) + \big[\Omega^1_r(p)(Df_p)\big](E_j) \qquad \blacksquare\end{aligned}$$

Theorem 5.2. Relative to $r \in \hat{G}$, let m_r be the smallest (necessarily nonnegative) number in $\mathrm{Spec}(\Delta_r)$ (i.e., the smallest mass^2 of particles coming from r). Then $b_r^2 \leqslant B_r m_r + Y_r\sqrt{m_r}$ or equivalently, $\sqrt{m_r} \geqslant \frac{1}{2}B_r^{-1}[(Y_r^2 + 4b_r^2 B_r)^{1/2} - Y_r]$. In the event the Yang–Mills equation holds (i.e., $\delta\Omega = 0$), we obtain $m_r \geqslant b_r^2/B_r$.

Proof. Let $f \in C(P, W_r)$ with $\Delta_r f = m_r f$ and $\| f \|_r = 1$. Then

$$\begin{aligned}b_r^2 = b_r^2 \| f \|_r^2 &\leqslant \textstyle\int_{Ph} |\Omega_r(p)(f(p))|_r^2 \mu(p) = ((r' \circ \Omega)(f), (r' \circ \Omega)(f))\\ &= (D_r D_r f, (r' \circ \Omega)(f)) = (D_r f, \delta_r[(r' \circ \Omega)(f)])\\ &= \int_P \langle D_r f_p, \Omega^1_r(p)(D_r f_p)\rangle \mu(p) + \int_P \langle D_r f_p, (\delta\Omega)_r(p)(f(p))\rangle \mu(p)\\ &\qquad\qquad \text{(by Lemma 5.1)}\\ &\leqslant \int_P |D_r f_p| \cdot |\Omega^1_r(p)(D_r f_p)| \mu(p) + \int_P |D_r f_p| \cdot |(\delta\Omega)_r(p)(f(p))| \mu(p)\\ &\leqslant B_r \| D_r f \|^2 + Y_r \| D_r f \| \| f \| = B_r m_r + Y_r \sqrt{m_r}\end{aligned}$$

where we have used the definitions of b_r, B_r, and Y_r, the Cauchy–Schwarz inequality, Lemma 3.5, and Corollary 3.4. ■

6. ADDITIONAL COMMENTS AND QUESTIONS

(A) The case where $G=U(1)\equiv\{e^{i\theta}|\theta\in\mathbb{R}\}$ (e.g., electromagnetism) deserves special consideration. For each integer k, let $\hat{k}$: $U(1)\rightarrow U(\mathbb{C})$ $[=U(1)]$ be the representation $\hat{k}(e^{i\theta})=e^{ik\theta}I$. The map $\Psi_{\hat{k}}$: $\mathrm{Hom}(\mathbb{C},\mathbb{C})\rightarrow C^{\infty}(U(1),\mathbb{C})$ is given by $\Psi_{\hat{k}}(zI)(e^{i\theta})=\mathrm{tr}(\hat{k}(e^{i\theta})\circ zI)=ze^{ik\theta}$. Since the eigenspaces of $\Delta_{U(1)}=-D_{\theta}^{2}$ are all of the form $\Psi_{\hat{k}}(\mathrm{Hom}(\mathbb{C},\mathbb{C}))\oplus\Psi_{-\hat{k}}(\mathrm{Hom}(\mathbb{C},\mathbb{C}))$, it follows from the Peter–Weyl theorem that $\{\hat{k}|k\in\mathbb{Z}\}$ is a complete set of mutually inequivalent unitary representations of $U(1)$.

The closed subgroups of $U(1)$ are all finite and cyclic; the one of order k is $\mathbb{Z}_{k}\equiv\{\exp(i2\pi m/k)|m=1,2,\ldots,k\}$. If the holonomy group of ω at $p\in P$ is $\mathbb{Z}_{N}$, then $0\in\mathrm{Spec}(\Delta_{\hat{k}})$ for all k which are multiples of N, since k^{2} will then be an eigenvalue of the Laplace operator on $U(1)/\mathbb{Z}_{N}$ and Theorem 4.1 applies. Conversely, if $0\in\mathrm{Spec}(\Delta_{\hat{k}})$ for some $k\neq0$, then by Theorem 4.2, $k^{2}\in\mathrm{Spec}[U(1)/G_{0}']$ and so the holonomy group is $\mathbb{Z}_{N}$ for some N dividing k.

Using the notation of Section 5, let $B\equiv B_{\hat{1}}$, $b\equiv b_{\hat{1}}$, and $Y\equiv Y_{\hat{1}}$. Since the Lie algebra homomorphism $\hat{k}'$ is just $k\hat{1}'$, we have $B_{\hat{k}}=|k|B$, $b_{\hat{k}}=|k|b$ and $Y_{\hat{k}}=|k|Y$. Consequently, Theorem 5.2 yields $|k|^{2}b\leqslant|k|Bm_{\hat{k}}+|k|Y\sqrt{m_{\hat{k}}}$ or $|k|b<Bm_{\hat{k}}+Y\sqrt{m_{\hat{k}}}$. This not only implies that $m_{\hat{k}}>0$ for $k\neq0$, but also $m_{\hat{k}}\rightarrow\infty$ as $|k|\rightarrow\infty$ provided $b>0$. Consequently, if $b>0$, then $\min\{m_{\hat{k}}|\,|k|\neq0\}$ exists and is positive. Interestingly, we have proved in Bleecker (1982) that the property $b>0$ is generic if $\dim M\geqslant4$. Hence, it is not surprising that all electrically charged particles seem to have mass no less than some fixed positive number.

(B) An intriguing question is whether the characteristic numbers of the principal bundle $P\rightarrow M$ can be determined from $\mathrm{Spec}(\Delta)$ or $\mathrm{Spec}(\Delta_{r})$ for various $r\in\hat{G}$. The terms of the asymptotic expansion of $\mathrm{trace}(e^{-t\Delta})$ (see Gilkey, 1975) will yield some information such as the total scalar curvature of (P,h), but characteristic numbers may be difficult to determine. Suppose P is trivial, say $P=M\times G$, and h is the product metric tensor $h_{M}\times k_{G}$. Then $\mathrm{Spec}(\Delta)=\{\lambda_{j}(M)+c_{r}|\lambda_{j}(M)\in\mathrm{Spec}(\Delta_{M})$ and $r\in\hat{G}\}$, and it follows that $\mathrm{Spec}(\Delta_{r})=\mathrm{Spec}(\Delta_{M})$, independent of $r\in\hat{G}$. In the general case, if r_{0} is the trivial unitary representation, then $\mathrm{Spec}(\Delta_{r_{0}})=\mathrm{Spec}(\Delta_{M})$. Thus, if it happens that $\mathrm{Spec}(\Delta_{r})$ is independent of r, then necessarily $\mathrm{Spec}(\Delta_{r})=\mathrm{Spec}(\Delta_{M})$ for all $r\in\hat{G}$. In particular, $0\in\mathrm{Spec}(\Delta_{r})$ and Theorem 4.2 implies $c_{r}\in\mathrm{Spec}(G/G_{0}')$ for all $r\in\hat{G}$, whence $\mathrm{Spec}(G/G_{0}')=\mathrm{Spec}(G)$. If we assume that the multiplicity of 0 in $\mathrm{Spec}(\Delta_{r})$ is d_{r} for all $r\in\hat{G}$, we can conclude (using Theorems 4.2 and 2.3) that Q^{*}: $L^{2}(G/G_{0}')\rightarrow L^{2}(G)$ is an isomorphism, whence $G_{0}'=\{e\}=G_{0}$ and P would then be a product bundle with product metric. Without any assumptions on multiplicities, it might still be possible to prove that the independence of Spec(Δ_r) on r implies P and h are products, but we leave this prospect to the interested reader.

(C) The discussion in (B) shows that when $P = M \times G$ with a product metric, all particles share a common mass^2 spectrum regardless of the representation. In the general case, Spec(Δ_r) will depend on r. We expect nature to favor those particles with representations r for which $m_r \equiv \min \mathrm{Spec}(\Delta_r)$ is small; it takes less energy to make particles with less mass. Thus, in view of Theorem 3.3, we see that particles coming from a representation r should be comparatively prevalent if there is an eigenspace $V(\lambda_i)$ of Δ on $C^\infty(P, \mathbb{C})$ which decomposes in such a way that r is a subrepresentation and $\lambda_i - c_r$ is comparatively small. Since the eigenvalues of Δ can vary widely depending on h_M and ω, we see that the populations of elementary particles may be dictated in an incalculable (yet theoretically precise) manner by the geometry of (P, h). In order to appreciate the difficulty in computing eigenvalues or eigenspaces even in fairly simple circumstances, the reader is invited to compute Spec(Δ_r) for arbitrary $r \in \hat{G}$ in the case where ω is a self-dual Yang–Mills field for a principal G-bundle of given index over S^4.

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